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Local conservation laws for the two-dimensional periodic SU($n+1$) Toda lattices

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Abstract. An infinite numnber of local conservation laws are derived for the two-dimensional periodic SU($n+1$) Toda lattice equations following the well known method for the sine–Gordon equation.

This is a short communication, which is a sequel to our previous paper (Farwell and Minami 1982b, to be referred to as II), where we demonstrated that the two-dimensional equations for the periodic SU($n+1$) Toda lattices are equivalent to two first-order differential equations, called the Kac–van Moerbeke (KVM) equations. Paper II naturally arose as a consequence of our earlier work on the two-dimensional Toda lattices governed by classical simple groups (Farwell and Minami 1982a to be referred to as I).

The existence of the KVM equations produces, in addition to the Bäcklund transformations discussed in II, a formal way of deriving an infinite series of local conservation laws. It is the method of deriving the conservation laws which we consider here. It is orthodox in the sense that it follows the well known method for the sine–Gordon equation. We can apply this approach, since, as was explained in II, the two-dimensional periodic SU($n+1$) Toda lattice equations are generalisations of the hyperbolic sine–Gordon equation. For a review on conservation laws, in particular for the sine–Gordon equation, we refer the reader to Chau's paper (Chau Wang 1980), which contains further useful references.

Firstly, we shall provide a resumé of the mathematical techniques and results developed in II and which we shall require here.

We define $\pi^+=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $\mu\equiv-\alpha_{n+1}$ to be the set of positive simple roots and the maximal root of the classical Lie algebra, $\mathbf{a}_n$, respectively. Then the $(n+1)\times(n+1)$ matrix K with entries

$$K_{ij}=2\alpha_i\alpha_j/(\alpha_j)^2 \qquad i,j=1,2,\ldots,n+1$$

is the extended Cartan matrix defined by (2.2) in II. K determines the Euclidean Lie algebra $\mathbf{a}_n^{(1)}$ (Helgason 1978), which is associated with periodic SU($n+1$) Toda lattices. Its $3(n+1)$ canonical generators, $E_{+\alpha}$, $E_{-\alpha}$ and H_α, satisfy for $\alpha,\beta\in\bar{\pi}=\pi^+\cup(-\mu)$,

$$\begin{aligned}&[H_\alpha,H_\beta]=0, \qquad [H_\alpha,E_{\pm\beta}]=\pm K_{\beta\alpha}E_{\pm\alpha}, \qquad [E_\alpha,E_{-\beta}]=\delta_{\alpha\beta}H_\alpha\\ &(adE_{\pm\alpha})^{1-K_{\beta\alpha}}(E_{\pm\beta})=0 \qquad \alpha\neq\beta\end{aligned} \tag{1}$$

(Serre 1966). If we use the representations for $E_{\pm\alpha}$, H_α, $\alpha \in \pi^+$ and the ordering of the positive simple roots given in I, then

$$H_{\alpha_{n+1}} = e_{n+1n+1} - e_{11} \qquad E_{+\alpha_{n+1}} = -e_{n+11}, \qquad E_{-\alpha_{n+1}} = -e_{1n+1}. \tag{2}$$

In terms of these generators, we can define gauge potentials, B_μ and B_π, as

$$B_u = B^{\mathrm{e}}_u + B^{\mathrm{h}}_u \qquad B_{\bar{u}} = B^{\mathrm{e}}_{\bar{u}} + B^{\mathrm{h}}_{\bar{u}} \tag{3}$$

where

$$B^{\mathrm{e}}_u = \sum_{\alpha\in\tilde{\pi}} \mathrm{e}^{\tilde{\sigma}_\alpha} E_{-\alpha} \qquad B^{\mathrm{e}}_{\bar{u}} = -\sum_{\alpha\in\tilde{\pi}} \mathrm{e}^{\sigma_\alpha} E_{+\alpha} \tag{4}$$

$$B^{\mathrm{h}}_u = \sum_{\alpha\in\tilde{\pi}} (\partial_u \psi_\alpha) H_\alpha \qquad B^{\mathrm{h}}_{\bar{u}} = -\sum_{\alpha\in\tilde{\pi}} (\partial_{\bar{u}} \tilde{\psi}_\alpha) H_\alpha. \tag{5}$$

Here we have used

$$\sigma_\alpha = -\sum_{\beta\in\tilde{\pi}} K_{\alpha\beta}\psi_\beta \qquad \tilde{\sigma}_\alpha = -\sum_{\beta\in\tilde{\pi}} K_{\alpha\beta}\tilde{\psi}_\beta \tag{6}$$

which automatically satisfy

$$\sum_{\alpha\in\tilde{\pi}} \sigma_\alpha = 0, \qquad \sum_{\alpha\in\tilde{\pi}} \tilde{\sigma}_\alpha = 0. \tag{7}$$

From the zero field-strength condition it may be inferred that B^{e}_u, B^{h}_u, $\mathrm{B}^{\mathrm{e}}_{\bar{u}}$ and $B^{\mathrm{h}}_{\bar{u}}$ satisfy the relations

$$\partial_{\bar{u}} B^{\mathrm{e}}_u = [B^{\mathrm{e}}_u, B^{\mathrm{h}}_{\bar{u}}] \tag{8a}$$

$$\partial_u B^{\mathrm{e}}_{\bar{u}} = [B^{\mathrm{e}}_{\bar{u}}, B^{\mathrm{h}}_u] \tag{8b}$$

$$\partial_{\bar{u}} B^{\mathrm{h}}_u - \partial_u B^{\mathrm{h}}_{\bar{u}} - [B^{\mathrm{e}}_u, B^{\mathrm{e}}_{\bar{u}}] = 0. \tag{9}$$

The equations (8) and (9) are just the subsidiary and main equations of I and hence the latter gives the periodic SU$(n+1)$ Toda lattice equations, namely

$$\partial_u \partial_{\bar{u}} (\sigma_\alpha + \tilde{\sigma}_\alpha) = -\sum_{\beta\in\tilde{\pi}} K_{\alpha\beta} \exp(\sigma_\beta + \tilde{\sigma}_\beta).$$

We now introduce a cyclic permutation matrix E defined by

$$E = \sum_{\alpha\in\tilde{\pi}} E_{+\alpha}. \tag{10}$$

This matrix has a similar effect to the one introduced in §5 of II except that now $E^{n+1} = -I$. The introduction of E enables us to rewrite the KVM equations in the form

$$[B^{\mathrm{h}}_u, E^{-1}] = \gamma^{-1}(B^{\mathrm{e}}_u - E^{-1} B^{\mathrm{e}}_u E) \tag{11a}$$

$$[B^{\mathrm{h}}_{\bar{u}}, E] = \gamma(B^{\mathrm{e}}_{\bar{u}} - E B^{\mathrm{e}}_{\bar{u}} E^{-1}) \tag{11b}$$

where γ is the parameter of the Lie transformation (Goursat 1925). It is straightforward to show that the KVM equations (11) are equivalent to the Toda lattice equations (9).

The first stage in the derivation of the local conservation laws involves the definition of two currents satisfying a continuity-type equation. We propose to use the currents defined by

$$j_u = \mathrm{Tr}(EB^{\mathrm{e}}_u), \qquad j_{\bar{u}} = -\mathrm{Tr}(E^{-1}B^{\mathrm{e}}_{\bar{u}}). \tag{12}$$

Since, by using equations (8) and (11),

$$\gamma^{-1}E\partial_{\bar{u}}B^{\mathrm{e}}_u = \gamma^{-1}[EB^{\mathrm{e}}_u, B^{\mathrm{h}}_{\bar{u}}] + (B^{\mathrm{e}}_{\bar{u}}B^{\mathrm{e}}_u - EB^{\mathrm{e}}_{\bar{u}}E^{-1}B^{\mathrm{e}}_u)$$

and

$$\gamma E^{-1}\partial_u B^{\mathrm{e}}_{\bar{u}} = \gamma[E^{-1}B^{\mathrm{e}}_{\bar{u}}, B^{\mathrm{h}}_u] + (B^{\mathrm{e}}_u B^{\mathrm{e}}_{\bar{u}} - E^{-1}B^{\mathrm{e}}_u E B^{\mathrm{e}}_{\bar{u}})$$

their difference satisfies

$$\gamma^{-1}E\partial_{\bar{u}}B^{\mathrm{e}}_u - \gamma E^{-1}\partial_u B^{\mathrm{e}}_{\bar{u}} = \gamma^{-1}[EB^{\mathrm{e}}_u, B^{\mathrm{h}}_{\bar{u}}] - \gamma[E^{-1}B^{\mathrm{e}}_{\bar{u}}, B^{\mathrm{h}}_u] + [B^{\mathrm{e}}_{\bar{u}}, B^{\mathrm{e}}_u] + [E^{-1}B^{\mathrm{e}}_u, EB^{\mathrm{e}}_{\bar{u}}]. \tag{13}$$

By taking the trace of both sides of equation (13) and introducing j_u and $j_{\bar{u}}$ from (12), we see that the currents satisfy the continuity equation

$$\gamma^{-1}\partial_{\bar{u}}j_u + \gamma\partial_u j_{\bar{u}} = 0. \tag{14}$$

Substituting for B^{e}_u, $B^{\mathrm{e}}_{\bar{u}}$ and E from (4) and (10) respectively, we find that j_u and $j_{\bar{u}}$ are explicitly given by

$$j_u = \sum_{i=1}^{n+1} \exp\tilde{\sigma}_i \qquad j\bar{u} = \sum_{i=1}^{n+1} \exp\sigma_i \tag{15}$$

where here and in the following we abbreviate σ_α with $\alpha = \alpha_i$ by σ_i and let the index i run modulo $(n+1)$.

Now, an infinite set of conservation laws is derivable from (14). Firstly, we note from (11*b*) that at $\gamma = 0$, $[B^{\mathrm{h}}_{\bar{u}}, E]$ vanishes and hence $\tilde{\sigma}_i = 0$, $i = 1, 2, \ldots, n+1$. So near $\gamma = 0$, we expand $\tilde{\sigma}_i$ in powers of γ as follows

$$\tilde{\sigma}_i = \sum_{k=1}^{\infty} \gamma^k c_i^{(k)}. \tag{16}$$

In terms of Toda's displacement variables introduced in (4.7) of II

$$\sigma_i = q^{\mathrm{B}}_{i-1} - q_i, \qquad \tilde{\sigma}_i = q_i - q^{\mathrm{B}}_i$$

and hence using the expansion (16) we may write

$$q^{\mathrm{B}}_i = q_i - \sum_{k=1}^{\infty} \gamma^k c_i^{(k)} \tag{17}$$

which itself implies

$$\sigma_i = q_{i-1} - q_i - \sum_{k=1}^{\infty} \gamma^k c_{i-1}^{(k)}. \tag{18}$$

By substituting (16) and (18) in (11*a*) and equating coefficients of powers of γ, we obtain equations relating $c_i^{(k)}$ for different values of k, for example

$$\partial_u(q_{i-1} - q_i) = c_i^{(1)} - c_{i-1}^{(1)} \tag{19a}$$

$$\partial_u c_i^{(1)} = c_i^{(2)} - c_{i+1}^{(2)} + \tfrac{1}{2}[(c_i^{(1)})^2 - (c_{i+1}^{(1)})^2] \tag{19b}$$

$$\partial_u c_i^{(2)} = c_i^{(3)} - c_{i+1}^{(3)} + (c_i^{(2)}c_i^{(1)} - c_{i+1}^{(2)}c_{i+1}^{(1)}) + \tfrac{1}{6}[(c_i^{(1)})^3 - (c_{i+1}^{(1)})^3] \tag{19c}$$

and so on. Moreover from (7) and (16), we note that

$$\sum_{i=1}^{n+1} c_i^{(k)} = 0 \tag{20}$$

for all values of k.

On the other hand, if we substitute (16) and (18) into the continuity equation (14), then, again by equating coefficients of powers of γ, we obtain an infinite series of continuity equations. The first few examples are

$$\partial_u \sum_{i=1}^{n+1} \exp(q_{i-1}-q_i) + \tfrac{1}{2}\partial_{\bar{u}} \sum_{i=1}^{n+1} (c_i^{(1)})^2 = 0 \tag{21a}$$

$$\partial_u \sum_{i=1}^{n+1} (c_{i-1}^{(1)} \exp(q_{i-1}-q_i)) - \partial_{\bar{u}} \sum_{i=1}^{n+1} c_i^{(1)}[c_i^{(2)} + \tfrac{1}{6}(c_i^{(1)})^2] = 0 \tag{21b}$$

$$\partial_u \sum_{i=1}^{n+1} \exp(q_{i-1}-q_i)[c_{i-1}^{(2)} - \tfrac{1}{2}(c_i^{(1)})^2] - \partial_{\bar{u}} \sum_{i=1}^{n+1} [\tfrac{1}{2}(c_i^{(2)})^2 + c_i^{(3)}c_i^{(1)} + \tfrac{1}{2}(c_i^{(1)})^2 c_i^{(2)}] = 0, \tag{21c}$$

where we have used the conditions (20).

Obviously we may write the continuity equations (21) in terms of the q_i only by using the constraint equations (19). In particular, if we choose

$$c_i^{(1)} = -\partial_u q_i \tag{22a}$$

as a solution of (19a), then the $c_i^{(k)}$, $k \geqslant 2$, are related to q_i by higher and higher orders of differentiation. For instance, from (19b)

$$c_i^{(2)} - c_{n+1}^{(2)} = -\sum_{j=i}^{n} \partial_u^2 q_j + \tfrac{1}{2}(\partial_u q_i)^2 - \tfrac{1}{2}(\partial_u q_{n+1})^2 \tag{22b}$$

where, from (20)

$$c_{n+1}^{(2)} = -\sum_{i=1}^{n} c_i^{(2)}.$$

Consequently (21a) may easily be rewritten as

$$\partial_u \sum_{i=1}^{n+1} \exp(q_{i-1}-q_i) + \tfrac{1}{2}\partial_{\bar{u}} \sum_{i=1}^{n+1} (\partial_u q_i)^2 = 0, \tag{23a}$$

(21b) becomes

$$\partial_u \sum_{i=1}^{n+1} [(\partial_u q_{i-1}^{(1)})\exp(q_{i-1}-q_i)] + \partial_{\bar{u}}\left[\sum_{i=1}^{n} (\partial_u q_i)\partial_u^2\left(\sum_{j=1}^{n} q_j\right) + \tfrac{1}{3}\sum_{i=1}^{n+1} (\partial_u q_i)^3\right] = 0 \tag{23b}$$

and so on.

By using the difference displacement variable

$$\rho_i = q_{i-1} - q_i$$

we may re-express (21a) without using the particular solution (22a). From (19a), we find that

$$\sum_{j=0}^{n-1}\left(\sum_{k=j}^{n-1} \partial_u \rho_{i+k+2}\right)^2 = \sum_{j=1}^{n+1} (c_j^{(1)})^2 + (n+1)(c_i^{(1)})^2 \tag{24}$$

and, hence by summing (24) from $i = 1$ to $n+1$ and substituting in (21a), we obtain

$$\partial_u \sum_{i=1}^{n+1} e^{\rho_i} + \frac{1}{4(n+1)} \partial_{\bar{u}} \sum_{i=1}^{n+1} \sum_{j=0}^{n-1} \left(\sum_{k=j}^{n-1} \partial_u \rho_{i+k+2} \right)^2 = 0. \tag{25}$$

We shall consider this last equation explicitly for the algebra $\mathbf{a}_1^{(1)}$, which is associated with the periodic SU(2) Toda lattice. Since in this case $\rho_1 = -\rho_2$, (25) becomes simply

$$2\partial_u \cosh \rho_1 + \tfrac{1}{4}\partial_{\bar{u}} (\partial_u \rho_1)^2 = 0. \tag{26}$$

This is a well known continuity equation, since it is readily derivable from the sinh–Gordon equation by multiplying by $\partial_u \rho$.

We showed in II that it is possible to obtain from the KVM equation two combinations of variables , ρ_i and ρ_i^{B}, both of which staisfy the periodic SU($n+1$) Toda lattice equation. Hence we could use

$$\sigma_i = q_{i-1}^{\mathrm{B}} - q_i^{\mathrm{B}} - \sum_{k=1}^{\infty} c_i^{(k)} \gamma^k$$

as an alternative to (18). In the resultant working, q_i^{B} and q_{i-1}^{B} replace q_i and q_{i-1} respectively in (19a) and the left-hand sides of (19b) and (19c) are replaced by $\partial_u c_{i+1}^{(1)}$ and $\partial_u c_{i+1}^{(2)}$ respectively. Obviously then ρ_i in (25), for example, is replaced by ρ_i^{B}.

It is apparent that we could also carry out the derivation of the conservation laws by expanding σ_i, $i = 1, 2, \ldots, n+1$ near infinity in inverse powers of γ. However, the subsequent infinite series of conservation laws can be obtained simply by interchanging u and $\bar{u}$ in (23), because the system is 'dual' symmetric under the exchange

$$(\gamma, u, \bar{u}) \to (\gamma^{-1}, \bar{u}, u).$$

Furthermore, we can suggest another formal way of obtaining a set of conserved currents, by recalling that the Zakharov–Shabat equation

$$\partial_{\bar{u}} B_u - \partial_u B_{\bar{u}} - [B_u, B_{\bar{u}}] = 0, \tag{27}$$

i.e. equations (8) and (9), are just the compatibility condition of the two equations

$$\partial_u M = [M, B_u], \qquad \partial_{\bar{u}} M = [M, B_{\bar{u}}]. \tag{28}$$

Then, by using (8) and (28), we can show that for n being a positive integer

$$\partial_u (B_{\bar{u}}^{\mathrm{e}} M^n) + \partial_{\bar{u}} (B_u^{\mathrm{e}} M^n) = [B_{\bar{u}}^{\mathrm{e}} M^n, B_u] + [B_u^{\mathrm{e}} M^n, B_{\bar{u}}]$$

and hence it is obvious that

$$\mathrm{Tr}[\partial_u (B_{\bar{u}}^{\mathrm{e}} M^n) + \partial_{\bar{u}} (B_u^{\mathrm{e}} M^n)] = 0. \tag{29}$$

Consequently, we may define a set of currents

$$\tilde{J}_u^{(n)} = \mathrm{Tr}(B_u^{\mathrm{e}} M^n), \qquad \tilde{J}_{\bar{u}}^{(n)} = \mathrm{Tr}(B_{\bar{u}}^{\mathrm{e}} M^n) \qquad n = 1, 2, 3, \ldots \tag{30}$$

which, by virtue of (29), satisfies the continuity equation

$$\partial_{\bar{u}} \tilde{J}_u^{(n)} + \partial_u \tilde{J}_{\bar{u}}^{(n)} = 0.$$

The conservation laws (23) may be reduced to one dimension by making the identification $u = \bar{u} \equiv t$. As in § 4 of I†, if we reduce B_u and $B_{\bar{u}}$ given by (3) to A

† We should like to point out here that $L(0)$ in (4.7)–(4.11) of I should be read as $L_{(0)}$, because $L(t)$ does not necessarily reduce to $RL(t)R^{-1}$ as $t \to 0$. $L_{(0)}$ is to be defined by $L_{(0)} \equiv R^{-1}L(t)R^{-1}$.

and B respectively, then (27) becomes the Lax pair equation

$$\partial_t L = [L, A]$$

where $L = B - A$ and so is explicitly given by

$$L = -\sum_{\alpha\in\tilde{\pi}} (e^{\sigma_\alpha} E_\alpha + e^{\tilde{\sigma}_\alpha} E_{-\alpha}) - \sum_{\alpha\in\tilde{\pi}} \partial_t(\psi_\alpha + \tilde{\psi}_\alpha) H_\alpha. \tag{31}$$

As is well known, the conservation laws in the one-dimensional case are produced from

$$\partial_t \operatorname{Tr} L^p = 0 \qquad p = 1, 2, 3, \ldots .$$

The first non-trivial conserved current occurs for $p = 2$, when, from (31),

$$\operatorname{Tr} L^2 = \sum_{i=1}^{n+1} \{[\partial_t(\psi_i + \tilde{\psi}_i - \psi_{i-1} - \tilde{\psi}_{i-1})]^2 + 2\exp(\sigma_i + \tilde{\sigma}_i)\} \tag{32}$$

is a constant. By differentiating (32) with respect to t, we produce the same form of equation as from the reduction of (23a) (or equivalently (25)), since

$$\sigma_i + \tilde{\sigma}_i = q^{\mathrm{B}}_{i-1} - q^{\mathrm{B}}_i = \rho^{\mathrm{B}}$$

and

$$\partial_t(\psi_i + \tilde{\psi}_i - \psi_{i-1} - \tilde{\psi}_{i-1}) = -\partial_t q^{\mathrm{B}}_{i-1} = c^{(1)}_{i-1}.$$

However, it is not clear to us at present what are the roles of the parameter γ of the Lie transformation and of the 'dual symmetry' (or relativistic invariance) when the reduction to one dimension takes place.

The one-dimensional reduction presented in § 4 of I has recently beeen discussed by Aomoto in a different mathematical context (Aomoto 1982). His work may be relevant to ours since he considers a periodic system equivalent to the periodic Toda lattice in terms of an 'infinite-dimensional analogue' of the Iwasawa decomposition.

Finally, we should like to remark that there seem to be several different methods of obtaining infinite series of conservation laws; for example, that of Takhadzhyan (1974), used in the specific case of the sine–Gordon equation and of Mikhailov *et al* (1981) for two-dimensional Toda lattices. These different methods may combine to give insight into the general problem of conservation laws.

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